A SPECIAL CASE OF CONTACT BETWEEN AN ASYMMETRIC PUNCH AND AN ELASTIC HALF-SPACE

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A solution is obtained to a group of problems concerning the contact between an asymmetric punch and an elastic half-space. The special case is considered where the punch penetrates from a contact point along a rectilinear segment.

If a smooth rigid punch with a shape described by the equation

$$Z = f(\mathbf{r}),$$

is pressed into an elastic half-space Z > 0, then the contact pressure underneath p and the vertical displacement of the elastic material w in the process of punch penetration to a depth α are equal to the integrals

$$p(\mathbf{r},\ \alpha) = \int\limits_{\alpha_r}^{\alpha} p_0(\mathbf{r};\ [\Gamma_\lambda])\, d\lambda, \tag{1}$$

$$w(\mathbf{r},\ \alpha) = \int\limits_{0}^{\alpha} w_0(\mathbf{r};\ [\Gamma_\lambda])\, d\lambda. \tag{2}$$

Here p_0 and w_0 denote, respectively, the pressure and the displacement in the case of a single flat punch; $\mathbf{r}$ is the radius vector of point M($\mathbf{r}$) in the XY plane; α_r is the depth of penetration at which point M($\mathbf{r}$) falls into the contact zone; and line Γ_λ is generated by loading the family of contours within the contact zone.

Thus, p and w in (1) and (2) are expressed as a superposition of p_0 and w_0 for flat punches whose contours at every instant of time are the respective Γ_λ-curves; p_0 and w_0 are functions of α only indirectly through Γ_α and are functionals of those lines (which is denoted by square brackets). The contours Γ_μ must satisfy the nonlinear integral equation [1, 2]

$$\int\limits_{0}^{\mu} w_0(\mathbf{r};\ [\Gamma_\lambda])\, d\lambda = \mu - f(\mathbf{r}),\ \mathbf{r} \in \Gamma_\mu \tag{3}$$

at any $0 < \mu \leq \alpha$.

Let the family of contours Γ_a satisfy Eq. (3). Then penetration α is related to the new parameter a:

$$\alpha(a) = \frac{N'(a)}{Q'(a)}, \tag{4}$$

$$P(a) = \alpha(a)\ Q(a) - N(a), \tag{5}$$

where

$$Q(a) = \int\limits_{S_a} p_0(\mathbf{r};\ [\Gamma_a])\, d\sigma, \qquad N(a) = \int\limits_{S_a} p_0(\mathbf{r};\ [\Gamma_a])\, f(\mathbf{r})\, d\sigma, \tag{6}$$

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where $Q(a)$ is the total contact force on a flat punch.

We consider a punch whose sections by Z = const planes are confocal ellipses:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

with the major semiaxes a and the minor semiaxes $b = \sqrt{a^2 - l^2}$, $2l$ being the focal distance.

It will be shown that in this case the contours within the contact zone at the instant of loading are plane confocal ellipses which are completely defined by one parameter (e.g., by the major semiaxis), inasmuch as the focal distance has already been fixed. The problem is one-dimensional with respect to displacement, although the contact pressure remains three-dimensional, and its solution is constructed according to formulas (1), (2), (4)-(6).

The surface of this punch is described by the equation

$$Z = f[a(X, Y)].$$

By changing the parameter l and the form of function f, we obtain punches either axially symmetric ($l = 0$) or spatially symmetric ($l \neq 0$) with the initial contact along an ellipse or along a rectilinear segment $-l \leq X \leq l$, $Y = 0$. The displacement outside a flat elliptic punch [3]:

$$w_0(X, Y; a) = w_0(r, \theta; a) = \frac{1}{K\left(\frac{l}{a}\right)} F\left(\arcsin\frac{a}{r}, \frac{l}{a}\right)$$

remains constant along confocal ellipses. Here K and F are a complete and an incomplete elliptic integral, respectively; r and θ are elliptic coordinates related to Cartesian coordinates as follows:

$$\begin{cases} X = r\cos\theta, \\ Y = \sqrt{r^2 - l^2}\sin\theta, \end{cases} \quad \frac{D(X, Y)}{D(r, \theta)} = \frac{r^2 - l^2\cos^2\theta}{\sqrt{r^2 - l^2}}.$$

By inserting w_0 and f into (3), we ascertain that the equation is satisfied on confocal ellipses if it is satisfied on even only one point of the contour, regardless of the magnitude of α. Letting the major semiaxis of the contact ellipse a be the independent variable, we arrive at the one-dimensional linear Volterra equation with respect to $d\alpha(a)/da$:

$$\int_l^a [w_0(a; t) - 1]\frac{d\alpha(t)}{dt}\,dt = -f(a).$$

One can find the relation between α and a without solving this equation, however, by using the well-known expression for the pressure under a flat elliptic punch [3, 4]:

$$p_0(X, Y; a) = \frac{A}{K\left(\frac{l}{a}\right)\sqrt{1 - \frac{l^2}{a^2}}\sqrt{a^2 - X^2 - \frac{Y^2}{1 - \frac{l^2}{a^2}}}},$$

$$A = \frac{G}{1 - \nu}.$$

We have

$$Q(a) = \int_{S_a} p_0(X, Y; a)\,dXdY = \frac{2\pi A a}{K\left(\frac{l}{a}\right)},$$

$$N(a) = \int_{S_a} p_0(X, Y; a)\, f(X, Y)\,dXdY$$

$$= 4Aa^2\frac{E\left(\frac{l}{a}\right)}{K\left(\frac{l}{a}\right)}\int_l^a \frac{f(r)\,dr}{\sqrt{(a^2 - r^2)(r^2 - l^2)}} - 4A\int_l^a \sqrt{\frac{a^2 - r^2}{r^2 - l^2}}\, f(r)\,dr, \tag{7}$$

$$p(r,\ \theta;\ a)=\int\limits_r^a p_0(r,\ \theta;\ t)\ \frac{d\alpha(t)}{dt}\ dt=A\int\limits_r^a \frac{\frac{d\alpha(t)}{dt}\ tdt}{K\left(\frac{l}{t}\right)\sqrt{(t^2-r^2)(t^2-l^2\cos^2\theta)}}, \tag{8}$$

$$w(r,\ \theta;\ a)=\int\limits_l^a w_0(r,\ \theta;\ t)\ \frac{d\alpha(t)}{dt}\ dt=\int\limits_l^a \frac{\frac{d\alpha(t)}{dt}}{K\left(\frac{l}{t}\right)}\ F\left(\arcsin\frac{t}{r},\ \frac{l}{t}\right)tdt. \tag{9}$$

Here G is the shear modulus; ν is the Poisson ratio; and K, E are complete elliptic integrals. Formulas (4), (5), (7), (8), (9) yield the solution to the problem in quadratures.

In the special case of $l = 0$ we have

$$Q(a)=4Aa,\ N(a)=4A\int\limits_0^a \frac{f(r)\,rdr}{\sqrt{a^2-r^2}}. \tag{10}$$

From this follows the well-known solution [4, 5] for any punch with axial symmetry.

Expanding $\frac{d\alpha(t)}{dt}\cdot\frac{1}{K\left(\frac{l}{t}\right)\sqrt{t^2-l^2\cos^2\theta}}$ into a series at the point $t=a$ and retaining only the zeroth-order term, we will obtain the first term of the pressure asymptote (8) near the boundary $0 < (a-r)/a \ll 1$ at any a:

$$p(r,\ \theta;\ a)\approx\frac{A}{K\left(\frac{l}{a}\right)}\cdot\frac{d\alpha(a)}{da}\sqrt{\frac{a^2-r^2}{a^2-l^2\cos^2\theta}}. \tag{11}$$

When $a \gg 1$, then Q, N, α, and P asymptotically approach the case of axial symmetry (10). Pressure $p(r,\theta;a)$ is close to the pressure in the symmetric case when $a \geq r \geq l$.

In concluding the analysis of this solution, we note that contact problems with initial line contact are widely encountered in practice and have been treated by approximation in [4, 6, 7, 8]. As contact is made, the region involved usually becomes elongated. In our case the half-width of the contact ellipse $\Psi(X)$ changes as a function of X:

$$\Psi(X)=\frac{b}{a}\sqrt{a^2-X^2}.$$

The curvature of punch sections by X = const planes is at points on the X axis

$$\varkappa(X)=\begin{cases}\frac{1}{\sqrt{2}}f'\ [a(X,\ Y)]|_{Y=0}\ \frac{l}{l^2-X^2} & \text{for}\ X\leqslant l,\\ \frac{1}{\sqrt{2}}f'\ [a(X,\ Y)]|_{Y=0}\ \frac{X}{X^2-l^2} & \text{for}\ X\geqslant l.\end{cases}$$

As the Z = 0 plane is approached, function $f(a)$ behaves as $Z = f(X-1)$ and $Z = f(Y^2/2l)$ along the X axis and along the Y axis, respectively, indicating that the curve section along the Y axis is of a twice higher order.

We will now consider the case where integrals (4), (5), and (7) are expressed in terms of elliptic ones. Let $f = c[a(X, Y)-l]$. In the section by the Y = 0 plane we have

$$f[a(X,\ 0)]=\begin{cases}0 & \text{for}\ |X|\leqslant l,\\ c(|X|-l) & \text{for}\ |X|\geqslant l.\end{cases}$$

In the sections by X = const planes the punch becomes parabolic toward the X axis and the curvature on the X axis is

$$\varkappa(X)=\begin{cases}\dfrac{c}{\sqrt{2}}\cdot\dfrac{l}{l^2-X^2} & \text{for } X\leqslant l,\\[2ex] \dfrac{c}{\sqrt{2}}\cdot\dfrac{X}{X^2-l^2} & \text{for } X\geqslant l.\end{cases}$$

We introduce the dimensionless variables

$$\xi=\frac{a}{l},\ \eta=\frac{b}{l},\ \rho=\frac{r}{l},\ x=\frac{X}{l},\ y=\frac{Y}{l},\ \psi(x)=\frac{\Psi(X)}{l},$$

$$\alpha(\xi)=\frac{\alpha(a)}{cl},\ f(\rho)=\frac{f(r)}{cl},\ P(\xi)=\frac{P(a)}{Acl^2},$$

$$N(\xi)=\frac{N(a)}{Acl^2},\ Q(\xi)=\frac{Q(a)}{Al},\ p(\rho,\theta;\xi)=\frac{p(r,\theta;a)}{Ac}$$

With the integrals in (7) evaluated,

$$Q(\xi)=\frac{2\pi\xi}{K_1},\ N(\xi)=-4\xi\delta K_2+\pi[1+\xi^2(2\delta-1)]+4(K_2-E_2),$$

the solution to problem (4), (8) will be reduced to

$$\left.\begin{aligned}\alpha(\xi)&=K_1\left\{\xi\delta+\frac{2}{\pi}[K_2(1-\delta)-E_2]\right\},\\ P(\xi)&=\alpha(\xi)\,Q(\xi)-N(\xi),\\ p(\rho,\theta;\xi)&=\int\limits_{\rho}^{\xi}\frac{\dfrac{d\alpha(\tau)}{d\tau}\tau d\tau}{K\left(\dfrac{1}{\tau}\right)\sqrt{(\tau^2-\rho^2)(\tau^2-\cos^2\theta)}}.\end{aligned}\right\}\tag{12}$$

Here $K_1=K(1/\xi)$; $E_1=E(1/\xi)$; $\delta=E_1/K_1$; $K_2=K[\sqrt{(\xi^2-1)}/\xi]$; $E_2=E[\sqrt{(\xi^2-1)}/\xi]$; K and E are complete elliptic integrals.

Some results obtained with formulas (12) on a BÉSM-6 computer are shown in Figs. 1, 2, 3. The dashed-dotted lines represent the asymptotes for small contact zones ($\xi\sim 1$), the dashed lines represent the asymptotes for large contact zones ($\xi\gg 1$).

For a narrow contact zone $0<\xi-1\ll 1$ or $\xi\sim 1$, the asymptotes are

$$\left.\begin{aligned}\alpha(\xi)&=\frac{\xi-1}{2}\left(\ln\frac{8}{\xi-1}+1\right)+\frac{(\xi-1)^2}{4}\ln\frac{8}{\xi-1}+O[(\xi-1)^2],\\ \alpha'(\xi)&=\frac{\xi}{2}\ln\frac{8}{\xi-1}+O[(\xi-1)],\\ P(\xi)&=2\pi(\xi-1)+O\ (\xi-1)\quad.\end{aligned}\right\}\tag{13}$$

A comparison with solution (12) shows that these formulas are valid with a less than 5% error: for α within $1<\xi<1.6$, for α' within $1<\xi<1.14$, and for P within $1<\xi<1.14$.

When $\xi\gg 1$, the contact zone approaches a circle. We have here

$$\left.\begin{aligned}\alpha(\xi)&=\frac{\pi}{2}\xi-1-\frac{\pi}{8}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right),\\ \alpha'(\xi)&=\frac{\pi}{2}+\frac{\pi}{8}\cdot\frac{1}{\xi^2}+O\left(\frac{1}{\xi^4}\right),\\ P(\xi)&=\pi\xi^2-\pi+O\left(\frac{1}{\xi}\right).\end{aligned}\right\}\tag{14}$$

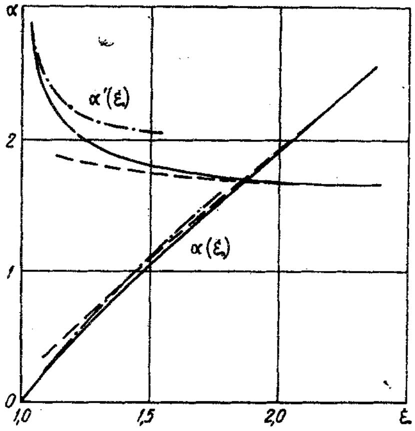

Fig. 1

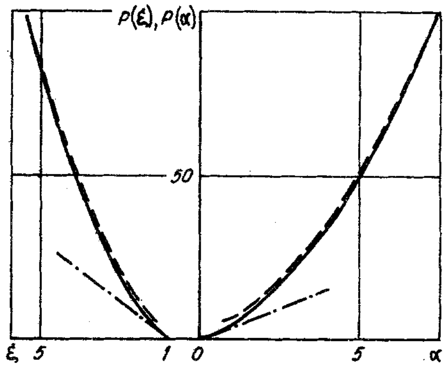

Fig. 2

Fig. 1. Punch penetration α and its derivative α' as functions of the major semiaxis of the contact ellipse ξ. Dashed lines represent the asymptotes for large values of ξ; dashed-dotted lines represent the asymptotes for small values of ξ. All quantities here and in Figs. 2 and 3 are dimensionless.

Fig. 2. Total contact force P as a function of the penetration depth α and of the major semiaxis of the contact ellipse ξ. Dashed lines represent the asymptotes for large values of ξ and α; dashed-dotted lines represent the asymptotes for small values of ξ and α.

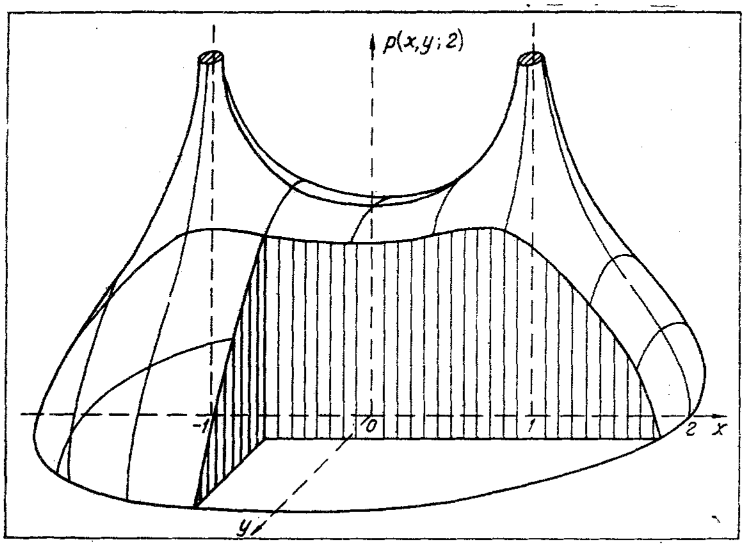

Fig. 3. Distribution of contact pressure p for $\xi = 2$.

With a less than 5% error, these formulas apply to the range $\xi > 1.35$ for α, $\xi > 1.4$ for α', and $\xi > 1.15$ for P. Inserting the asymptotes to $\alpha'(\xi)$ and K_1 into the contact-pressure integral (12), we obtain the same expression for both $\xi \sim 1$ and $\xi \gtrsim \rho \gg 1$:

$$p(\rho,\ \theta;\ \xi) \approx L = \ln \frac{\sqrt{\xi^2 - \rho^2} + \sqrt{\xi^2 - \cos^2\theta}}{\sqrt{\rho^2 - \cos^2\theta}}. \tag{15}$$

One must emphasize the extraordinary closeness of the pressure asymptotes for extreme values of ξ. Furthermore, formula (15) deviates from the exact solution by less than 3% along the entire contact zone for all values of ξ.

When $\xi \gtrsim \rho \gg 1$, the pressure tends toward the solution for the case with axial symmetry [9]:

$$p(\rho,\ \theta;\ \xi) \approx L \approx \operatorname{Arch} \frac{\xi}{\rho}.$$

Expression (15) for pressure p simplifies on the axes of the ellipse. On axis y ($x = 0$, $\theta = \pi/2$, $y = \sqrt{\rho^2 - 1}$) we have

$$p(0, y; \xi) \approx \ln\left(\frac{\xi}{\rho} + \frac{1}{\rho}\sqrt{\xi^2 - \rho^2}\right) = \operatorname{Arch}\frac{\xi}{\rho} \approx \begin{cases} \sqrt{\eta^2 - y^2} & \text{at } \xi \sim 1, \\ \dfrac{\sqrt{\eta^2 - y^2}}{\sqrt{y^2 + 1}} & \text{at } \xi > y \gg 1, \\ \ln\dfrac{2\xi}{\sqrt{y^2 + 1}} & \text{at } \xi \gg 1, y \ll 1. \end{cases}$$

On axis x ($y = 0$, $x = \rho$ at $x > 1$; $x = \cos\theta$, $\rho = 1$ at $x < 1$) we have

$$p(x, 0; \xi) \approx \ln\frac{\sqrt{\xi^2 - x^2} + \sqrt{\xi^2 - 1}}{\sqrt{|x^2 - 1|}} \approx \begin{cases} \dfrac{\eta}{\sqrt{1 - x^2}} & \text{at } \xi \sim 1, x \ll 1, \\ \operatorname{Arch}\dfrac{\xi}{x} \approx \dfrac{\sqrt{\xi^2 - x^2}}{x} & \text{at } \xi \gg x \gg 1, \\ \ln\dfrac{2\xi}{\sqrt{1 - x^2}} & \text{at } \xi \gg 1, x \ll 1. \end{cases}$$

At points $x = \pm 1$ on the x-axis the pressure is unbounded. Finally, near the y-axis the contact pressure is given by the simple formula:

$$p(x, y; \xi) \approx \frac{\sqrt{\psi^2(x) - y^2}}{\sqrt{1 - x^2}} \quad \text{at } \xi \sim 1, x \ll 1, \tag{16}$$

where

$$\psi(x) = \frac{\eta}{\xi}\sqrt{\xi^2 - x^2} \approx \eta\left(1 - \frac{x^2}{2\xi^2}\right).$$

Up to $\xi = 1.25$, formula (16) yields the answer in asymptotic terms with a less than 10% error at $x < 0.5$ and less than 25% at $x < 0.7$.

NOTATION

r	is the radius;
ρ	is the dimensionless radius;
θ	is the angle in elliptic coordinates;
X, Y, Z	are the Cartesian coordinates;
x, y	are the dimensionless Cartesian coordinates;
w	is the displacement of elastic material under a punch of any shape;
p	is the pressure under a punch of any shape;
w_0	is the displacement of elastic material under a flat punch;
p_0	is the pressure under a flat punch;
f	is the function describing the punch shape;
P	is the total contact force under a punch of any shape;
Q	is the total contact pressure under a flat punch;
N	is the component of total force;
α, μ, λ	are the depths of punch penetration;
S_α	is the contact area;
Γ_α	is the contact contour;
M	is the point;
K, E	are the complete elliptic integrals;
F	is the incomplete elliptic integral;
a, b	are the major and minor semiaxes of ellipses;
ξ, η	are the dimensionless major and minor semiaxes of ellipses;
l	is half the focal distance;
D	is the differentiation operator;
A, c	are the coefficients in the expressions for function f and pressure under a flat punch;
G	is the shear modulus of the elastic half-space;

ν is the Poisson ratio for the elastic half-space;

Ψ, ψ are the ordinate and dimensionless ordinate of the points on an ellipse;

L is the logarithmic approximation of pressure distribution;

$\varkappa$ is the curvature.

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